

Linear Second-Order Differential Equations

$$ay'' + by' + cy = f(t) \quad \begin{aligned} &= \text{Linear Second-Order} \\ &\quad \text{constant-coefficient} \\ &\quad \text{Differential Equation} \\ (a, b, c \text{ constants}, a \neq 0) \end{aligned}$$

Aim : Find a general solution.

Homogeneous case : $f(t) = \text{zero function}$, ie $ay'' + by' + cy = 0$

Observations (Homogeneous Case)

- 1/ zero function is solution
 - 2/ y_1, y_2 solutions $\Rightarrow y_1 + y_2$ solutions
 - 3/ y_1 solution, λ real number $\Rightarrow \lambda y_1$ solution
- \Rightarrow Solutions to $ay'' + by' + cy = 0$ are a subspace of the vector space of twice differentiable functions

Theorem Fix any t_0 in \mathbb{R} .

$$\begin{aligned} T : \left\{ \begin{array}{l} \text{Solutions to} \\ ay'' + by' + cy \end{array} \right\} &\longrightarrow \mathbb{R}^2 && \text{called initial conditions} \\ y(t) &\longrightarrow \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} && \text{for the solution} \end{aligned}$$

is a one-to-one, onto, linear transformation.
 ↗ uniqueness ↙ existence

Consequence : $\left\{ \begin{array}{l} \text{Solutions to} \\ ay'' + by' + cy \end{array} \right\}$ is 2 dimensional.

We need to find two linearly independent solutions to $ay'' + by' + cy$.

Let's try and guess a solution. Try $y = e^{rt}$ for some constant r

$$\Rightarrow y' = re^{rt}, \quad y'' = r^2 e^{rt} \quad \text{auxiliary polynomial.}$$

$$\Rightarrow ay'' + by' + cy = (ar^2 + br + c)e^{rt}$$

If $ar^2 + br + c = 0 \Rightarrow y = e^{rt}$ is a solution to $ay'' + by' + cy = 0$
auxiliary equation

Three Cases :

1/ Two distinct real solutions r_1, r_2 . ($b^2 - 4ac > 0$)

$$r_1 \neq r_2 \Rightarrow \{e^{r_1 t}, e^{r_2 t}\} \text{ L.I.}$$

$\Rightarrow \{e^{r_1 t}, e^{r_2 t}\}$ basis for $\{ \text{solutions to } ay'' + by' + cy = 0 \}$

$$\Rightarrow \text{General Solution to } ay'' + by' + cy = 0 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2/ One repeated real solution r_1 ($b^2 - 4ac = 0$)

$$\text{Consider } y = te^{r_1 t} \quad r_1 = \frac{-b}{2a}$$

$$\Rightarrow y' = e^{r_1 t} + r_1 t e^{r_1 t}$$

$$\Rightarrow y'' = 2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t}$$

$$+ (2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t})$$

$$\Rightarrow ay'' + by' + cy = +b(e^{r_1 t} + r_1 t e^{r_1 t}) \\ + c(te^{r_1 t})$$

$$= (2ar_1 + b)e^{r_1 t} + (ar_1^2 + br_1 + c)te^{r_1 t} = 0$$

$$\{e^{r_1 t}, te^{r_1 t}\} \text{ L.I.} \Rightarrow$$

$$\text{General Solution to } ay'' + by' + cy = 0 = c_1 e^{r_1 t} + c_2 te^{r_1 t}$$

3/ Two complex conjugate non-real zeroes $\alpha \pm \beta i$ ($b^2 - 4ac < 0$)

$$e^{(\alpha+i\beta)t} = e^{\alpha t} \cdot e^{\beta t i} = e^{\alpha t} (\cos(\beta t) + i \sin(\beta t)) \\ = e^{\alpha t} \cos(\beta t) + i e^{\alpha t} \sin(\beta t)$$

Complex valued solution

Want real valued solution

Fact : $\{e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\}$ L.I. real-valued solutions

$$\Rightarrow \text{General Solution to } ay'' + by' + cy = 0 = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

Example $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$

$$\Rightarrow \text{General Solution} = c_1 \cos(t) + c_2 \sin(t)$$

Non-homogeneous Case : $ay'' + by' + cy = f(t)$ not the zero function

Fix y_p a particular solution. Let y be another solution

$$\Rightarrow ay'' + by' + cy = f(t) = ay_p'' + by_p' + cy_p$$

$$\Rightarrow a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = 0$$

$$\Rightarrow y - y_p = y_h \quad \leftarrow \begin{matrix} \text{solution to} \\ \text{homogeneous problem} \end{matrix}$$

$$\Rightarrow y = y_h + y_p \quad \begin{matrix} \text{General solution} \\ \text{to homogeneous problem} \end{matrix}$$

Conclusion : 1, General solution

$$\text{to } ay'' + by' + cy = f(t) = y_h + y_p \quad \begin{matrix} \downarrow \\ \text{fixed} \\ \text{particular} \\ \text{solution} \end{matrix}$$

$$2/ \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} = \begin{pmatrix} y_p(t_0) \\ y_p'(t_0) \end{pmatrix} \Rightarrow y = y_p \quad \left(\begin{matrix} y_h(t_0) = 0 \\ y_h'(t_0) = 0 \end{matrix} \Rightarrow y_h = \text{zero function} \right)$$

Problem: For general $f(t)$ there is no easy way to find a particular solution.

Method of Undetermined Coefficients

To find a particular solution to $\text{degree of polynomial}$

$$ay'' + by' + cy = P_m(t) e^{kt}$$

$$\text{try } y_p(t) = t^s (A_0 + A_1 t + \dots + A_n t^n) e^{kt}$$

If k is not a root of auxiliary equation set $s=0$

If k is a simple root of auxiliary equation set $s=1$

If k is a repeated root of auxiliary equation set $s=2$

To find a particular solution to

$$ay'' + by' + cy = \begin{cases} P_m(t) e^{kt} \cos(kt) \\ P_m(t) e^{kt} \sin(kt) \end{cases}$$

$$\text{try } y_p(t) = t^s (A_0 + A_1 t + \dots + A_n t^n) e^{kt} \cos(kt)$$

$$+ t^s (B_0 + B_1 t + \dots + B_n t^n) e^{kt} \sin(kt)$$

If $k+i\ell$ is not a root of auxiliary equation set $s=0$

If $k+i\ell$ is a root of auxiliary equation set $s=1$

Remark The aim is to find suitable coefficients $A_0, \dots, A_n, B_0, \dots, B_n$.

This will lead us to solving a linear system.

Example Find a general solution to $y'' - 2y' + y = t^2 - 5t + 5$

a) Auxiliary Equation : $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

$\Rightarrow 1$ is a repeated real root

\Rightarrow General solution to $y'' - 2y' + y = 0$ is $c_1 e^t + c_2 t e^t$

b) Let $y_p(t) = A_0 + A_1 t + A_2 t^2$

$$\Rightarrow y_p'(t) = A_1 + 2A_2 t$$

$$y_p''(t) = 2A_2$$

$$\begin{aligned} \Rightarrow y_p'' - 2y_p' + y_p &= 2A_2 - 2(A_1 + 2A_2 t) + A_0 + A_1 t + A_2 t^2 \\ &= (A_0 - 2A_1 + 2A_2) + (A_1 - 4A_2)t + A_2 t^2 \end{aligned}$$

\Rightarrow Need to solve

$$A_0 - 2A_1 + 2A_2 = 5$$

$$A_1 - 4A_2 = -5$$

$$A_2 = ?$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right) \xrightarrow{\quad} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right) \xrightarrow{\quad} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

$$\Rightarrow A_0 = 1, A_1 = -1, A_2 = 1$$

$$\Rightarrow$$
 General solution to $y'' - 2y' + y = t^2 - 5t + 5$

$$\text{is } c_1 e^t + c_2 t e^t + t^2 - t + 1$$

Example Find a solution to $y'' - 2y' + y = (5t+2)e^t$

with initial conditions $y(0) = 1, y'(0) = 2$.

$$y_p(t) = t^2 (A_0 + A_1 t) e^t \quad \leftarrow 1 \text{ is a repeated root}$$

$$= (A_0 t^2 + A_1 t^3) e^t$$

$$\Rightarrow y_p'(t) = (2A_0 t + 3A_1 t^2) e^t + (A_0 t^2 + A_1 t^3) e^t$$

$$= (2A_0 t + (3A_1 + A_0) t^2 + A_1 t^3) e^t$$

$$\Rightarrow y_p''(t) = (2A_0 + (6A_1 + 2A_0) t + 3A_1 t^2) e^t$$

$$+ (2A_0 t + (3A_1 + A_0) t^2 + A_1 t^3) e^t$$

$$= (2A_0 + (6A_1 + 4A_0) t + (6A_1 + A_0) t^2 + A_1 t^3) e^t$$

$$\Rightarrow y_p'' - 2y_p' + y_p = (2A_0 + (6A_1 + 4A_0) t + (6A_1 + A_0) t^2 + A_1 t^3) e^t$$

$$- 2 (2A_0 t + (3A_1 + A_0) t^2 + A_1 t^3) e^t$$

$$+ (A_0 t^2 + A_1 t^3) e^t$$

$$= (2A_0 + 6A_1 t) e^t$$

$$\text{Need } (2A_0 + 6A_1 t) e^t = (2+6t) e^t \Rightarrow A_0 = 1 = A_1$$

$$\Rightarrow \text{General solution to } y(t) = C_1 e^t + C_2 t e^t + (t^2 + t^3) e^t$$

$$y'' - 2y' + y = (2+6t) e^t$$

$$y(0) = C_1 = 1 \Rightarrow C_1 = C_2 = 1$$

$$y'(0) = C_1 + C_2 = 1$$

$$\Rightarrow y(t) = e^t + t e^t + t^2 e^t + t^3 e^t \text{ is solution}$$

Superposition Principle

$$ay_1'' + by_1' + cy_1 = f_1(t)$$
$$ay_2'' + by_2' + cy_2 = f_2(t) \Rightarrow y_1 + y_2 \text{ solution to } ay'' + by' + cy = f_1(t) + f_2(t)$$

Overview & Strategy to solve $ay'' + by' + cy = f(t)$

- 1 Find general solution to $ay'' + by' + cy = 0$ using auxiliary equation $ar^2 + br + c = 0$
- 2 Break up $f(t)$ into sum of $P_m(t)e^{kt}$, $P_m(t)e^{kt}\cos(lt)$, $P_m(t)e^{kt}\sin(lt)$.
- 3 Find particular solutions for each piece using method of undetermined coefficients.
- 4 Sum general homogeneous solution and all particular solutions
- 5 Given initial conditions, calculate $y(t_0)$ and $y'(t_0)$ and solve for C_1 and C_2 .